Applied Geostatistics (G17604-01)

- Midterm Examination -

Student ID: **Name:**

Announcements

- Submit your solution as *.pdf or *.word file before the class scheduled on May 2, 2018.

- Follow the format that gives a name to your report or report file:
  (mid)-(student ID)-(last name)
  For example, Seoyoon Kwon's filename must be mid-181ERG01-Kwon.

- Any discussion is **NOT** allowed in this take-home examination.

- Fill your name in the following:
  "I, _________________, swear I solve all problems in this midterm examination by myself. I will take any disadvantages if any dishonesty such as cheating is acted on my solution."

Problem 1. [10 pts.]

1-1. Let N be a positive integer and let f be the function defined by:

$$f(x) = \begin{cases} \dfrac{2x}{N(N+1)}, & x = 1, 2, \dots, N \\ 0, & \text{elsewhere} \end{cases}$$

Show that f is a discrete density and find its mean.

Hint:

$$\sum_{x=1}^{N} x = \frac{N(N+1)}{2} \text{ and } \sum_{x=1}^{N} x^2 = \frac{N(N+1)(2N+1)}{6}$$

1-2. The Cauchy cumulative distribution function is

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x), \qquad -\infty < x < \infty$$

i. Show that this is a cdf.
ii. Find the density function.
iii. Find x such that $P(X > x) = 0.1$.

Problem 2. [10 pts.]

2-1. Let $X_1, \ldots, X_n$ be independent random variables having a common density with mean μ and variance σ^2. Set $\bar{X} = (X_1 + X_2 + \cdots, X_n)/n$.

a. By writing $X_k - \bar{X} = (X_k - \mu) - (\bar{X} - \mu)$, show that

$$\sum_{k=1}^{n} (X_k - \bar{X})^2 = \sum_{k=1}^{n} (X_k - \mu)^2 - n(\bar{X} - \mu)^2.$$

b. Conclude from (i) that

$$E\left\{\sum_{k=1}^{n} (X_k - \bar{X})^2\right\} = (\mathrm{n} - 1)\sigma^2.$$

2-2. Let X be a nonnegative continuous RV having density f and distribution function F. Show that X has finite expectation if and only if

$$\int_0^\infty \big(1 - F(x)\big)dx < \infty$$

and consequently

$$E\{X\} = \int_0^\infty \big(1 - F(x)\big)dx.$$

2-3. Let X_1, X_2, and X_3 be independent RV having finite positive variances σ_1^2, σ_2^2, and σ_3^2, respectively. Find the correlation between $X_1 - X_2$ and $X_2 + X_3$.

Problem 3. [10 pts.]

3-1. Let X and $Y = X^2$ be positive continuous random variables having densities f and g, respectively. Express f in terms of g and g in terms of f.

3-2. Let X be uniformly distributed on (0, 1). Find the density of $Y = X^{1/\beta}$, where $\beta \neq 0$.

3-3. Let X be a positive continuous random variable having density f.

What is the density of $Y = \frac{1}{1+X}$?

3-4. The Weibull cumulative distribution function is

$$F(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, x \geq 0, \alpha > 0, \beta > 0.$$

a. Find the density function.
b. Show that if W follows a Weibull distribution, then $X = \left(\frac{W}{\alpha}\right)^{\beta}$ follows an exponential distribution.
c. How could outcomes from a Weibull distribution be generated using a random number generator?

Problem 4. [10 pts.]

Consider the following data pertaining to temperature in a domain (e.g., reservoir and surface):

Observation number	Value
1	12
2	13.5
3	13
4	15.3
5	14.1
6	9.1
7	7.6
8	17.2
9	16.1
10	11
11	13.5
12	16
13	15.2
14	8.6
15	9.1
16	11.5
17	14.2
18	14.9
19	16.3
20	8.4
21	17.8
22	16.5
23	16.9
24	12.5
25	13.4
26	9.8
27	8.9
28	10.3
29	10.8
30	12.4

Discuss the graphical procedure to transform the above data to Gaussian distributed values with unit mean and variance. Implement that procedure in MS-EXCEL.

Problem 5. [10 pts.]

5-1. Let $f(x, y) = xe^{-x(y+1)}$, $0 \le x < \infty$, $0 \le y < \infty$

a. Find the marginal densities of X and Y. Are X and Y independent?
b. Find the conditional densities of X and Y.

5-2. If X_1 is uniform on [0, 1], and, conditional on X_1, X_2 is uniform on [0, X_1], find the joint and marginal distributions of X_1 and X_2.

5-3. Let $f(x, y) = e^{-y}$, $0 \le x \le y$

a. Find cov(X, Y) and the correlation between X and Y.
b. Find the marginal densities
c. Find the conditional densities.
d. Find $E(X \mid Y = y)$ and $E(Y \mid X = x)$.
e. Find the density functions of $E(X \mid Y)$ and $E(Y \mid X)$.

Problem 6. [10 pts.]

6-1. Show that if the joint distribution of X_1 and X_2 is bivariate normal, then the joint distribution of $Y_1 = a_1X_1 + b_1$ and $Y_2 = a_2X_2 + b_2$ is bivariate normal.

6-2. Let be independent standard normal random variables. Show that the joint distribution of

$Y_1 = a_{11}X_1 + a_{12}X_2 + b_1$

$Y_2 = a_{21}X_1 + a_{22}X_2 + b_2$

is bivariate normal.

6-3. Consider the bivariate distribution of x and y defined as follows: let u and v be independent N(0, 1) random variables. Set $x = u$ if $u \cdot v \geq 0$ while $x = -u$ if $u \cdot v < 0$, and set $y = v$.

Show that

a. x and y are each N(0, 1), but their joint distribution is not bivariate normal;
b. x^2 and y^2 are statistically independent, but x and y are not.

Problem 7. [10 pts.]

7-1. A variable X is characterized by the following probability density function:

$$f_X(x) = kx; \quad \text{for } x \in [0, 1]$$
$$= 0 \quad \text{everywhere else}$$

a. For what value of *k* is the above function a legitimate pdf?
b. What is the cdf value corresponding to:

   x = 0.7

   x = 0.3

   Can you say anything about the symmetry of the above distribution based on the above results?
c. What is the quantile $x_{0.75}$?
d. What is the expected value and variance of the above distribution?

Problem 8. [10 pts.]

8-1. Shafts manufactured for use in optical storage devices have diameters that are normally distributed with mean $\mu = 0.652$ cm and standard deviation $\sigma = 0.003$ cm. The specification for the shaft diameter is 0.650 ± 0.005 cm.

a. What proportion of shafts manufactured by this process meet the specifications?
b. The process mean can be adjusted through calibration. If the mean is set to 0.650 cm, what proportion of the shafts will meet the specification?
c. If the mean is set to 0.650 cm, what must the standard deviation be so that 99% of the shafts will meet the specification?

Problem 9. [10 pts.]

Given the following data for porosity in a reservoir:

0.16, 0.40, 0.18, 0.39, 0.13, 0.25, 0.19

Assuming a porosity range from 0.10 – 0.40 at steps of 0.05, compute the declustering weight you would assign to each datum.

Problem 10. [10 pts.]

10-1. Suppose that the two dimensional RF (X, Y) is uniformly distributed over the region $R = \{(x, y) \mid 0 < x < y < 1\}$.

a. What is the bivariate pdf?
b. What are the marginal distributions for X and Y?

10-2. Let X and Y have the joint density function

$f(x, y) = k(x - y)$, $0 \leq y \leq x \leq 1$ and 0 elsewhere.

a. Sketch the region over the density is positive.
b. Find k.
c. Find the marginal densities.
d. Find the conditional density of Y given X and X given Y.